Backreaction on Moving Mirrors

and Black Hole Radiation.

Niels Tuning and Herman Verlinde[+]

Institute for Theoretical Physics
University of Amsterdam
1018 XE Amsterdam
and
[+] *Joseph Henry Laboratories*
Princeton University
Princeton, NJ 08544

Abstract

We compute the effect of quantum mechanical backreaction on the spectrum of radiation in a dynamical moving mirror model, mimicking the effect of a gravitational collapse geometry. Our method is based on the use of a combined WKB and saddle-point approximation to implement energy conservation in the calculation of the Bogolyubov coefficients, in which we assume that the mirror particle has finite mass m. We compute the temperature of the produced radiation as a function of time and find that after a relatively short time, the temperature is reduced by a factor $1/2$ relative to the standard result. We comment on the application of this method to two-dimensional dilaton gravity with a reflecting boundary, and conclude that the WKB approximation quickly breaks down due to the appearance of naked singularities and/or white hole space-times for the relevant WKB-trajectories.

1. Introduction

Hawking discovered in 1975 that black holes emit particles due to quantum effects. The emission spectrum was found to be identical to that of a body with temperature $T = \frac{\kappa}{2\pi}$, where $\kappa = \frac{1}{4M}$ equals the surface gravity at the black hole horizon [1]. As a result of efforts to understand this effect, Unruh then showed that the Minkowski vacuum appears to be a thermal state at temperature $T = \frac{a}{2\pi}$ according to an observer with acceleration a, [2]. Davies and Fulling subsequently developed a simple *"moving mirror"* model, which provides a very useful analogy to the black hole situation [3, 4]. By choosing suitable trajectories, the moving mirror mimics many of the features of black hole radiance, with the difference that the quantum fields propagate in flat Minkowski space instead of complicated geometries with high curvatures. The trajectory of the moving mirror represents the origin of the coordinate system of the black hole geometry and both systems display event horizons, which is essential to get thermal out-going radiation.

The standard derivation of the Hawking effect is based on a semi-classical approximation in which one works on a fixed back ground geometry and in which interactions between the infalling matter and the out-going virtual particles are ignored. As a consequence, a number of basic physical principles are violated in this procedure, such as energy conservation and unitarity. Backreaction effects however could give corrections to the standard semi-classical result. In this paper we will treat backreaction effects in the simple model of the moving mirror, due to the reflection of quantum fields off the boundary particle. We will use a WKB-type approximation.* Contrary to common expectations, we find that the backreaction has substantial, measurable effects on the (temperature of) the out-going radiation.

We will begin with a short review the calculation of the out-going radiation of a mirror with a specified trajectory, which corresponds to a fixed background geometry in the black hole case. We do this by the method of saddlepoints, and find the same result as was originally derived in [3]. Then we take the rest mass of the mirror *finite* and calculate classically the back reaction effect of the reflection of massless scalar fields off the boundary. Again by using a saddlepoint method we obtain the out-going radiation, which will again be thermal radiation, but this time with half the temperature.

We then comment on the application of the same method to two-dimensional dilaton gravity with a reflecting boundary. Here we find that the WKB approximation quickly breaks down because the relevant WKB trajectories that should determine the out-going spectrum necessarily contain naked singularities and/or white hole space-times.
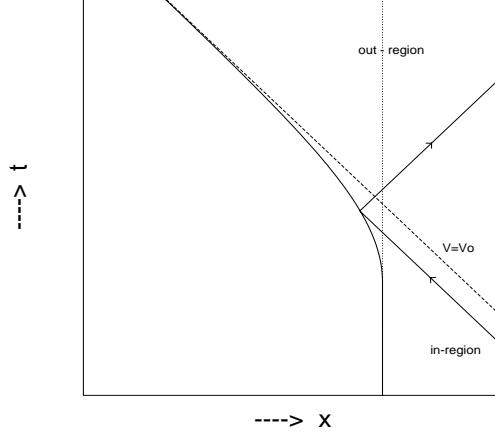
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*For recent related papers see [5, 6, 8].

Figure 1: The mirror trajectory $z(t)$ is asymptotic to the null ray $v = v_0$.

2. Fixed Mirror Trajectory

Consider a reflecting boundary in 1+1 dimensional Minkowski space, following the trajectory

$$x = z(t). \tag{2.1}$$

and a massless scalarfield, satisfying the Klein-Gordon equation

$$\Box \phi = \partial_u \partial_v \phi = 0 \tag{2.2}$$

with $u = x + t$ and $v = x - t$. We impose the reflecting boundary condition that the field has to vanish on the boundary,

$$\phi(t, z(t)) = 0. \tag{2.3}$$

The mirror trajectory that mimics the effect of the gravitational collapse geometry is given by

$$z(t) = -t - \frac{1}{\kappa} e^{-2\kappa(t-v_0)} + v_0 \tag{2.4}$$

where κ is a constant (corresponding to the surface gravity at the horizon) and $v = v_0$ is the asymptote. The velocity of the mirror point approaches the speed of light exponentially fast

$$\mathrm{v}(t) \equiv \frac{dz(t)}{dt} = -1 + 2e^{-2\kappa(t-v_0)} \tag{2.5}$$

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Left-moving null rays after $v = v_0$ do not intersect the mirror trajectory. In the black hole analogy, these correspond to light-like trajectories that enter into the black hole region.

Now following the standard procedure to compute the particle spectrum emitted by this moving mirror, we consider the process of a mono-chromatic in-wave $\phi_\omega^{in} \sim e^{-i\omega v}$ from past null infinity $\mathcal{I}^-$ that is being reflected at the boundary. A late out-going null ray $u = \bar{u}$ intersects the mirror at time $\bar{t} = \frac{1}{2}(\bar{u} + v_0)$. Thus, given the trajectory $z(t)$, we get that a reflected light-particle along $u = \bar{u}$ originated from an incoming particle along the null-ray $v = \bar{v}$ with

$$\bar{v} = p(\bar{u})$$

where

$$p(\bar{u}) = v_0 - \frac{1}{\kappa} e^{-\kappa(\bar{u} - v_0)}. \tag{2.6}$$

The monochromatic in-wave thus corresponds to a reflected out-going wave of the form

$$\phi_\omega^{out}(u) \sim e^{-i\omega p(u)}$$

Note, however, that this reflected wave represents only half of the final wave-function, since the other half at $v > v_0$ will never reach the mirror and will thus disappear into the left asymptotic region corresponding to the black hole horizon.

The Bogolyubov transformations (see also [5, 4]) give the relations between the different expansions $\phi_{\omega'}^{out}$ and ϕ_ω^{in} of the scalar field. The Bogolyubov coefficients are computed by evaluating the overlap between the two-types of wave modes

$$\alpha_{\omega\omega'} = (\phi_{\omega'}, \phi_\omega), \qquad \beta_{\omega\omega'} = -(\phi_{\omega'}, \phi_\omega^*). \tag{2.7}$$

One obtains

$$\alpha_{\omega\omega'} = \frac{i}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int du \, e^{i\omega' u - i\omega p(u)}$$

$$\beta_{\omega\omega'} = -\frac{i}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int du \, e^{i\omega' u + i\omega p(u)}. \tag{2.8}$$

These integrals can be done exactly, but for later comparison we will use a saddlepoint method. The saddle points for $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ lie respectively at $u = \bar{u}_\pm$ with

$$\bar{u}_\pm = v_0 - \frac{1}{\kappa} \log(\pm \frac{\omega'}{\omega}) \tag{2.9}$$

The saddlepoint $u = \bar{u}_+$ that contributes to $\alpha_{\omega\omega'}$ corresponds to the physical reflection time, which is uniquely determined for given in and out-frequency via the Doppler relation

$$\omega' = \omega \frac{1 + \text{v}}{1 - \text{v}} \tag{2.10}$$

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with v the velocity of the mirror given in (2.5). The saddlepoint $u = \bar{u}_-$ that contributes to $\beta_{\omega\omega'}$, on the other hand, has an imaginary part equal to π/κ, and thus corresponds to the virtual reflection process that gives rise to the particle creation phenomenon. These out-going reflection times correspond to ingoing times $\bar{v}_\pm$ with

$$\bar{v}_\pm = v_0 \mp \frac{\omega'}{\kappa\omega} \tag{2.11}$$

Hence for the Bogolyubov coefficients we find

$$\alpha_{\omega\omega'} \approx e^{i\omega'\bar{u}_+ - i\omega\bar{v}_+} \approx e^{-\frac{i\omega'}{\kappa}\ln(\frac{\omega'}{\omega}) - iv_0(\omega - \omega') + \frac{i\omega'}{\kappa}}$$

$$\beta_{\omega\omega'} \approx e^{i\omega'\bar{u}_- + i\omega\bar{v}_-} \approx e^{-\frac{i\omega'}{\kappa}\ln(-\frac{\omega'}{\omega}) + iv_0(\omega + \omega') + \frac{i\omega'}{\kappa}}. \tag{2.12}$$

The out-going radiation spectrum is given by

$$F(\omega') = \sum_\omega |\beta_{\omega\omega'}|^2$$

So, knowing that $\sum_\omega(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = 1$ and that the ratio $|\alpha_{\omega\omega'}|^2/|\beta_{\omega\omega'}|^2$ is independent of ω, we get

$$F(\omega') = \frac{1}{|\frac{\alpha_{\omega\omega'}}{\beta_{\omega\omega'}}|^2 - 1} = \frac{1}{e^{2\pi\omega'/\kappa} - 1} \tag{2.13}$$

which is precisely the Planck spectrum for thermal (bosonic) radiation with temperature $k_B T = \frac{\kappa}{2\pi}$, [3].

3. Backreaction on the Mirror

The typical energy of the out-going particle radiation is of the order of κ (times $\hbar$). Note however that, given the outgoing energy ω', the energy ω of the incoming wave will become exponentially large at late times. This effect arises from the fact that the mirror remains accelerating for all times. However, energy conservation (which should also apply in the virtual scattering process used in computing the Bogolyubov coefficients) tells us that the total energy at early times can not be more then the total energy of the combined system at late times. It is easy to see that this implies that the above computation of the Bogolyubov coefficients breaks down at relatively early times in case the mirror has a finite mass.

We will therefore now take the mirror mass to be *finite* and impose energy conservation upon reflection. Consider (in 1+1 dimensions) a mirror particle with mass m and energy $\gamma(v) m$ which collides elastically with a photon with energy ω. The photon reemerges with a energy ω'. We again consider the reflection process and we want to express the energy ω' in terms of ω and the mirror velocity v just *after* the collision. Classically, one finds:

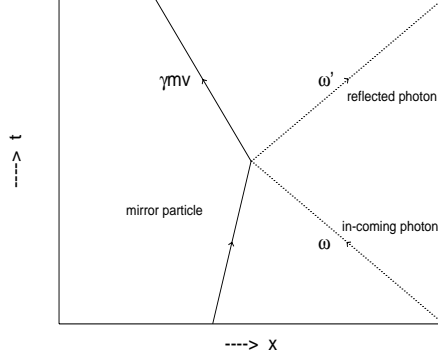$$\omega' = \omega \frac{1}{\delta^2 - \frac{2\omega}{m}\delta} \tag{3.1}$$

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Figure 2: The reflection of a photon off a mirror particle with finite mass.

with δ the Doppler factor

$$\delta = \sqrt{\frac{1-\mathrm{v}}{1+\mathrm{v}}}.$$

We now wish to consider the same classical mirror trajectory as in the previous section. We imagine therefore that there is some external force that acts on the mirror particle, which in the absence of other forces (such as those due to possible (virtual) collisions with the photons) will keep it in the given trajectory (2.4). More specifically, in determining the relevant classical WKB-trajectory, we keep the mirror trajectory *after* the collision with the photon fixed, so that it always remains asymptotic to $v = v_0$.[*] Hence the Doppler factor behaves in the out-going time u as before

$$\delta \approx e^{\frac{\kappa}{2}(u-v_0)} \approx \frac{1}{\sqrt{\kappa(v_0 - v)}}$$

Thus applying the general formula (3.1), we get the following relation between the in- and out-going energies in terms of the out-going time u

$$\omega' \approx \frac{\omega}{e^{\kappa(u-v_0)} - \frac{2\omega}{m}e^{\frac{\kappa}{2}(u-v_0)}} \tag{3.2}$$

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[*]The reason for this choice is that in the calculation of the out-going spectrum of radiation, one needs to investigate the action of an out-going annihilation operator on the final state. While this out-going mode is causally connected with the early part of the mirror trajectory, it is space-like separated from the late asymptotic trajectory. In other words, causality tells us that the mode *commutes* with the operators that measure the asymptotic mirror trajectory. (Note that this argument does in fact not hold in the black hole situation. In that case, the asymptotic time v_0 is dynamically determined in terms of the infalling matter forming the black hole, and therefore causally connected to the out-going radiation. See [6, 7]).

and in terms of the incoming time we have

$$\omega \approx \frac{\omega'}{\kappa(v_0 - v) + \frac{2\omega'}{m}\sqrt{\kappa(v_0 - v)}} \qquad (3.3)$$

Note that we get the old relation back in the limit $m \to \infty$. For a finite mirror mass, however, we notice in particular that the incoming energy always remains smaller than the kinetic energy of the mirror just after the collision. If we keep ω' fixed, then for late times ω approaches this maximal value

$$\omega \approx \frac{m}{2}e^{\frac{\kappa}{2}(u-v_0)} \approx \gamma m \qquad \text{for } u \to \infty \qquad (3.4)$$

since $\gamma \equiv \delta/(1 - v) \approx \delta/2$ for late times.

We would again like to use a combination of a WKB approximation for determining the relation between the in- and out-going wave packets and a saddlepoint approximation for the integrals in the expressions of the (corrected) Bogolyubov coefficients $\tilde{\alpha}_{\omega\omega'}$ and $\tilde{\beta}_{\omega\omega'}$. Following the same steps as in section 2, we can again express these coefficients in terms of the reflection points $(\tilde{u}_\pm, \tilde{v}_\pm)$ as

$$\tilde{\alpha}_{\omega\omega'} \approx e^{i\omega'\tilde{u}_+ - i\omega\tilde{v}_+}$$

$$\tilde{\beta}_{\omega\omega'} \approx e^{i\omega'\tilde{u}_- + i\omega\tilde{v}_-}. \qquad (3.5)$$

That is, the coefficients are just equal to the "phase jump" between the incoming and outgoing wave at the reflection point.

Using equation (3.2), we find the new saddlepoints $\tilde{u}_\pm$ at

$$\tilde{u}_\pm = v_0 - \frac{1}{\kappa}\log(\frac{\omega'}{\omega}) + \frac{2}{\kappa}\log(\sqrt{x^2 \pm 1} \pm x) \qquad (3.6)$$

while from (3.3) we obtain

$$\tilde{v}_\pm = v_0 - \frac{\omega'}{\kappa\omega}(\pm 1 + 2x^2 - 2x\sqrt{x^2 \pm 1}) \qquad (3.7)$$

with

$$x^2 = \frac{\omega\omega'}{m^2}.$$

As before, the physical reflection point $(\tilde{u}_+, \tilde{v}_+)$ that contributes to the Bogolyubov coefficient $\tilde{\alpha}_{\omega\omega'}$ is always real, while the saddle-point $(\bar{u}_-, \bar{v}_-)$ that contributes to $\tilde{\beta}_{\omega\omega'}$ again has an imaginary part. This imaginary part, however, is no longer constant, but depends on the variable x. Note further that the virtual reflection point $(\bar{u}_-, \bar{v}_-)$ is related to the real one by changing the sign of the *in-going* frequency ω. Indeed, the out-going radiation is produced because negative energy in-coming modes can propagate via virtual trajectories into positive energy out-going modes.
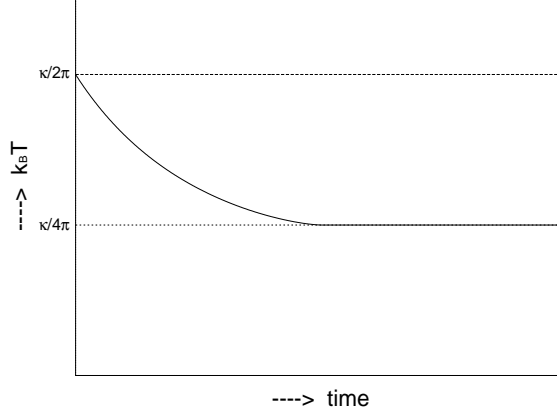
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Figure 3: The temperature of the radiation as a function of the out-going time $u = \frac{2}{\kappa} \log x^2$.

It is easy to see that, if we keep the out-going frequency ω' constant (which is still physically reasonable), the incoming frequency ω will at late times still grow exponentially in the out-going time, but now with half the e-folding time. This implies in particular that at late times (that is for $\kappa(\bar{u}_+ - v_0) \gg 1$), we have the relation

$$\tilde{u}_+ \approx \frac{2}{\kappa} \log \mathrm{x}^2 + \text{const.}$$

between the parameter x and the reflection time $\tilde{u}_+$. In the following we will make use of this relation to read off the time dependence of the radiation spectrum, given its expression in terms of the in and out-going frequencies.

A straightforward calculation now gives the following result for the new Bogolyubov coefficients:

$$\tilde{\alpha}_{\omega\omega'} \approx \alpha_{\omega\omega'} \, \exp \frac{2i\omega'}{\kappa} \Big[-\mathrm{x}(\sqrt{\mathrm{x}^2 + 1} - \mathrm{x}) + \log(\sqrt{\mathrm{x}^2 + 1} + \mathrm{x}) \Big] \tag{3.8}$$

and

$$\tilde{\beta}_{\omega\omega'} \approx \beta_{\omega\omega'} \, \exp \frac{2i\omega'}{\kappa} \Big[-\frac{i\pi}{2} + \mathrm{x}(\sqrt{\mathrm{x}^2 - 1} - \mathrm{x}) + \log(\sqrt{\mathrm{x}^2 - 1} - \mathrm{x}) \Big] \tag{3.9}$$

where $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ denote the uncorrected coefficients given in section 2. Note that we indeed get the old result back in the limit x$\to$ 0, which corresponds to the infinite mass limit $m \to \infty$.

For the ratio of the absolute values of the new coefficients we find

$$\frac{|\tilde{\alpha}_{\omega\omega'}|^2}{|\tilde{\beta}_{\omega\omega'}|^2} \approx \exp\Big[\frac{4\omega'}{\kappa} R(\mathrm{x}) \Big] \tag{3.10}$$

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with

$$R(x) = \pi - \arccos x + x\sqrt{1-x^2} \qquad \text{for } x < 1$$

$$= \pi \qquad\qquad\qquad \text{for } x > 1. \qquad (3.11)$$

We now define the out-going spectrum $F(\omega', x)$ as a function of the time parameter x as

$$F(\omega', x) = \frac{1}{\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} - 1} = \frac{1}{e^{\omega'/k_B T(x)} - 1} \qquad (3.12)$$

where we identified the time-dependent temperature as

$$k_B T(x) = \frac{\kappa}{4R(x)}. \qquad (3.13)$$

The temperature as a function of the outgoing time $\bar{u}$ is plotted in figure 3. We read off that after a relatively short amount of time of the order of

$$\bar{u} \approx v_0 + \frac{2}{\kappa} \log(\frac{2m}{\kappa}) \qquad (3.14)$$

(recall that the typical frequency of the out-going radiation is of the order of κ) the mirror radiates a constant flux of thermal radiation at a temperature $k_B T = \frac{\kappa}{4\pi}$.

In conclusion, we see that introducing a reflecting boundary with *finite* mass and imposing energy conservation upon reflection, leads to back reaction effects with dramatic consequences: the temperature of the mirror is half of the original result. This result could have been anticipated from equation (3.3). For late times, i.e. in the limit $v \to v_0$, the second term in the denominator determines the corrected Doppler relation between the in- and out-frequencies. In other words, the magnitude of the ingoing frequency for given ω' grows linearly with the Doppler factor δ, instead of quadratically.

4. DILATON GRAVITY

Over the past few years, much effort has gone into the study of two-dimensional dilaton gravity models of black holes [9]. We will now make a few comments about the application of the present WKB method to this model. We will not present any calculation, but refer for more details to [6, 9].

In the most concrete formulation of dilaton gravity, one imposes reflecting boundary conditions at some (arbitrarily chosen) critical value of the dilaton field. If one further chooses the matter fields to be massless scalars, one obtains a concrete correspondence with a dynamical moving mirror model. In this way, the model still shares many properties with the s-wave reduction of standard Einstein gravity, where the boundary corresponds to the
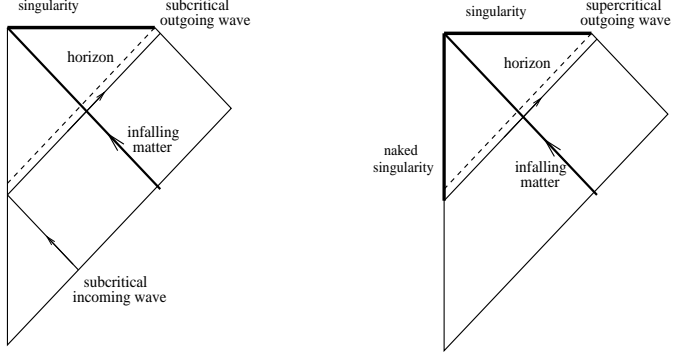
Figure 4: The classical space-times corresponding to sub-critical and super-critical out-going waves, resp. The classical WKB-trajectories that should give the (corrected) Bogolyubov coefficients for super-critical out-going waves involve space-times with naked singularities, and therefore do not lead to a definite prediction for the out-going spectrum.

$r = 0$ point. In particular, one finds that all incoming particle wave of energy below some critical value ω_{crit} will reflect back to infinity, provided it does not disappear behind the horizon of an already existing black hole. For larger incoming energies than the critical value ω_{crit}, the particle wave will never reflect back, but always lead to black hole formation.

In the sub-critical regime one can calculate the relation between the in- and out-going frequencies by solving for the corresponding classical geometry, see [6]. In the background geometry of a black hole of mass M, we send a signal with frequency ω' backwards in time from an outgoing time $\overline{u}$. It will bounce off the boundary, and produce an incoming signal at past infinity. The relation between the initial frequency and final frequency can be written for late times $\overline{u}$ as

$$\overline{u} = v_0 - \frac{1}{\lambda}\log(\frac{\omega'}{\omega}), \tag{4.1}$$

with λ the dilaton gravity "cosmological constant", which sets the scale of the surface gravity at the black hole horizon. The time v_0 is roughly the black hole formation time. The reflection off the boundary takes place at an ingoing time $\overline{v}$ given by

$$\overline{v} = v_0 + \frac{1}{\lambda}\log\left(\frac{\omega_{crit} - \omega}{\lambda}\right). \tag{4.2}$$

Somewhat surprisingly, we see in (4.1) that the backreaction of the geometry does not modify the old linearized relation between the in- and out-going frequencies. Hence, if we would use this relation to compute the Bogolyubov coefficients and the out-going spectrum, we find no

interesting corrections to the out-going spectrum. We have for $\omega < \omega_{crit}$

$$\tilde{\alpha}_{\omega\omega'} \approx e^{i\omega'\overline{u}_+ - i\omega\overline{v}_+} \approx e^{iv_0(\omega'-\omega) - \frac{i\omega'}{\lambda}\log(\frac{\omega'}{\omega}) - \frac{i\omega}{\lambda}\log((\omega_{crit}-\omega)/\lambda)}.$$

$$\tilde{\beta}_{\omega\omega'} \approx e^{i\omega'\tilde{u}_- + i\omega\tilde{v}_-} \approx e^{-\frac{\pi\omega'}{\lambda} + iv_0(\omega'+\omega) - \frac{i\omega'}{\lambda}\log(\frac{\omega'}{\omega}) + \frac{i\omega}{\lambda}\log((\omega_{crit}+\omega)/\lambda)} \tag{4.3}$$

So that

$$\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} = e^{\frac{2\pi\omega'}{\lambda}} \tag{4.4}$$

which seems to indicate that the out-going thermal spectrum receives no corrections whatsoever.

However, the result (4.1) for the classical reflection time is only valid in the sub-critical regime. As soon as

$$\omega > \omega_{crit} \tag{4.5}$$

we can no longer use the above formulas, because there no longer exists a classical trajectory that relates an outgoing wave of this frequency to a regular incoming wave. Indeed, as seen from (4.2), there is no longer a physical reflection time. Instead, the only classical solutions that contain out-going particles in the supercritical regime (4.5) are solutions that contain either white holes or naked singularities. In neither case, however, we know of good physical principles that provide concrete initial conditions for the out-going state. This super-critical regime (4.5) is reached very quickly, since in terms of the out-going frequency ω' the inequality reads

$$\omega' > \omega_{crit}e^{-\lambda(\overline{u}-v_0)}. \tag{4.6}$$

Thus we are forced to conclude that the WKB method breaks down when applied to two-dimensional dilaton gravity.

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